A Closed-Form Approximation to the Stochastic-Volatility

Jump-Diffusion Option Pricing Model

Eola Investments, LLC

www.eolainvestments.com

Rev0: May 2010

Rev1: October 2010
ABSTRACT

The risk-neutral pricing partial differential equation from a stochastic-volatility jump-diffusion model for a European call option is solved approximately using a regular perturbation and Green's function method. The closed-form approximate solution provides the same accuracy as the Heston model in in-sample and out-of-sample tests of S&P500 option pricing, with one to two orders of magnitude less computation time and much ease of numerical convergence.
INTRODUCTION

Empirical studies such as by Bates (1996) and Bakshi, Cao, and Chen (1997) (BCC) have shown that adding stochastic volatility and random jumps provide significant technical improvement over the theoretically acclaimed Black-Scholes and Merton option pricing model in practical applications. A common method to obtain an exact (semi) closed-form solution to the partial differential equation generated from a stochastic-volatility jump-diffusion process (SVJ) is to follow the characteristic function - Fourier inversion option pricing approach pioneered by Stein and Stein (1991) and Heston (1993). However, BCC have shown that the SV or SVJ model is still not correctly specified, since the parameters implied by the market option prices are not in quantitative agreement with those from time series data, and the volatility smirk is still there.

The parameters of a SVJ model are usually obtained by a calibration process using market data. The reasons are twofold: it is not easy to map a risk natural process from time series data to a risk neutral process from which option prices are derived (the market prices of risks still needs market data to quantify), and the option prices are determined by the expected future stock price distributions while time series data only provide historical distributions. The model calibration of the characteristic function type option pricing formula is still quite time consuming and, if not done optimally, prone to numerical instabilities as discussed by Kahl and Jäckel (2005), and Lord and Kahl (2007).

Alternatively, the stochastic volatility diffusion model (SV) is solved by singular perturbation method, see Fouque et. al. (2003) and Khasminskii and Yin (2005). Nevertheless, in such an approach a closed-form solution for a Poisson equation is yet to be found.

Since SV or SVJ is only an approximate model and its exact solution is not very convenient to use, it is worthwhile to find an approximate yet convenient and accurate option pricing formula from market practitioners' point of view. This is the aim for this paper. Section I describes the approximate solution for an European call option. Section II overview several option pricing models, including ad
hoc Black-Schole model, Cumulant matching model, Heston model, BCC model and Heston Nandi
GARCH model. Results of a limited comparative empirical study are given in Section III. Section IV
concludes the paper. Several integral formulas used in the derivation of the approximate solution are
provided in the Appendix.

I. An Approximate Solution To the Pricing PDE

Under risk-neutral measure, the underlying stock price, $S$, is assumed to follow a stochastic
volatility jump-diffusion process, and the return variance of the diffusion component, $v$, is assumed to
follow the CIR mean reverting process (BCC, 1997):

$$dS(t) = (r - \lambda \mu_J) S(t) dt + \sqrt{v(t)} S(t) \, dw_S(t) + J(t) S(t) \, dq(t)$$

(1)

$$dv(t) = \left[ \theta - \kappa v(t) \right] dt + \sigma_v \sqrt{v(t)} \, dw_v(t)$$

(2)

where $r$ is the risk-free interest rate; $\lambda$ is the frequency of jumps per year; $w_S(t)$ and $w_v(t)$ are
standard Brownian motions for underlying stock price and its diffusion part of return variance,
respectively, with covariance $\text{Cov}(dw_S, dw_v) = \rho \, dt$ ; $J(t)$ is the percentage jump size with mean
$\mu_J$ and standard deviation $\sigma_J$ ; $dq(t)$ is a Poisson Jump counter with the probability of jump
occurring be $\lambda dt$ ; $\kappa$, $\theta/\kappa$, and $\sigma_v$ are respectively the speed of mean reverting, long-run
mean, and variance of the return variance of the diffusion component. $q(t)$ and $J(t)$ are not correlated
with each other or with $w_S(t)$ and $w_v(t)$.

The pricing PDE is obtained by following the dynamic hedging argument of Black-Scholes and
the CAPM argument of Merton (1976) regarding jump-diffusion process:

$$\frac{1}{2} v S^2 \frac{\partial^2 C}{\partial S^2} + \left[ r - \lambda \mu_J \right] S \frac{\partial C}{\partial S} + \rho \sigma_v v S \frac{\partial^2 C}{\partial \nu \partial S} + \frac{1}{2} \sigma_v^2 v \frac{\partial^2 C}{\partial \nu^2} + (\theta - \kappa \nu) \frac{\partial C}{\partial \nu} + \frac{\partial C}{\partial \tau} - rC$$

$$+ \lambda E \left[ C(\tau, S(1+J), \nu) - C(\tau, S, \nu) \right] = 0$$

(3)

where $\tau$ is the time to expiration, $E[\cdot]$ is the expectation operator, and $C$ is the price of a European
call option with a strike price K and the boundary condition

\[ C(\tau=0, S, r, \nu) = \max(0, S - K) \]

To solve the PDE approximately, I intentionally introduce a “constant” parameter to represent the slow moving part of the stock return variance from diffusion process, \( \bar{\nu} \), in order to make the coefficients of the PDE constants, and introduce change of variables

\[ x = \ln \left( \frac{S}{K} \right), \quad -\infty < x < \infty \]

\[ u = \ln \left( \frac{\nu}{\bar{\nu}} \right), \quad -\infty < u < \infty \]

The PDE is transformed to

\[ L(C) = f_x(C) + f_u(C) \tag{4} \]

where \( L() \) is a linear operator

\[ L() = \frac{\partial}{\partial \tau} + a \frac{\partial}{\partial x} + b \frac{\partial^2}{\partial x^2} + c \frac{\partial}{\partial u} + d \frac{\partial^2}{\partial u^2} - \omega \frac{\partial^2}{\partial u \partial x} + (r - \delta) \times \]

\[ \delta = \lambda \mu_j \]

\[ a = -(r - \delta - \frac{\bar{\nu}}{2}) \]

\[ b = -\frac{\bar{\nu}}{2} \]

\[ c = \frac{\sigma^2}{2 \bar{\nu}} + \kappa - \frac{\theta}{\bar{\nu}} \]

\[ d = -\frac{\sigma^2}{2 \bar{\nu}} \]

\[ \omega = \rho \sigma \]

and

\[ f_x(C) = \frac{1}{2}(v - \bar{\nu}) \left( \frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x} + \frac{1}{2} \sigma^2 \left( \frac{1}{v} - \frac{1}{\bar{\nu}} \right) \right) \left( \frac{\partial^2 C}{\partial u^2} - \frac{\partial C}{\partial u} + \theta \frac{\partial C}{\partial u} \left( \frac{1}{v} - \frac{1}{\bar{\nu}} \right) \right) \]
\[ f(J, C) = \lambda E \left[ C(\tau, x + \ln(K) + \ln(1+J), v) - C(\tau, x + \ln(K), v) \right] - \delta C \]

and the boundary condition now is

\[ C(\tau=0, x, u) = h(x)(\exp(x) - 1)K \]

where \( h(x) \) is the Heaviside step function.

The causal Green's function of Eq. (4) is obtained by using the Fourier transformation technique and carrying out the complex integral by the Residual Theorem as (see Chen (2009) for details)

\[
G(\tau, \tau', x, x', u, u') = \frac{h(\tau - \tau') \exp(-r(\tau - \tau'))}{4 \sqrt{\pi^2 b (d - \omega^2/4b)(\tau - \tau')^2}} \times \exp \left[ \frac{[(x-x') - a(\tau - \tau')]^2}{4b(\tau - \tau')} + \frac{[(u-u') - c(\tau - \tau')]^2}{2b} + \frac{[(x-x') - a(\tau - \tau')] \omega^2}{4(b(\tau - \tau'))} \right] \]

(5)

The difficulty to solve the Eq. (4) lies in its heterogeneity, which can be circumvented by introducing a regular perturbation around the source terms as

\[ L(C) = \epsilon [f(J, C) + f(J, C)] \]

(6)

and assuming

\[ C \approx C_0 + \epsilon C_1 \]

(7)

Although in general an asymptotic series generated with regular perturbation method tends to diverge and it is indeed true in this case after first order perturbation, an empirical studies shown in section III that the first order perturbation already provides a good approximation for option pricing.

The perturbation form is obtained by inserting Eq. (7) into Eq. (6) and matching the terms with the same orders of \( \epsilon \):

Zeroth order \( O(0) \):

\[ L(C_0) = 0 \]

(8)

First order \( O(\epsilon) \):
\[ L(C_1) = f_v(C_0) + f_J(C_0) \quad (9) \]

Or separate \( C_1 \) into the SV contribution, \( C_{1v} \), and the jump contribution, \( C_{1J} \):

\[ L(C_{1v}) = f_v(C_0) \quad (9a) \]

\[ L(C_{1J}) = f_J(C_0) \quad (9b) \]

Eqs. [8], [9a], and [9b] are solved by employing the Green's function method:

\[ C_0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\tau, \tau' = 0, x, x', u, u') C(\tau' = 0, x'u') du' dx' \]

\[ C_0 = SN(d_1) - K \exp(-(r - \delta)\tau)N(d_2) \quad (10) \]

where

\[ d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \delta - \frac{\bar{v}}{2}\right)\tau}{\sqrt{\bar{v}\tau}} \]

\[ d_1 = d_2 + \sqrt{\bar{v}\tau} \]

and \( N(\cdot) \) is the standard normal cumulative distribution function. The zeroth order approximation takes the form of Black-Scholes formula and treats the jump as distributing dividend with a continuous dividend rate \( \delta \).

\[ C_{1v} = \int_{0}^{\infty} \int_{-\infty}^{\infty} G(\tau, \tau', x, x', u, u') f_v(C_0(\tau', x', u')) du' dx' d\tau' \]

\[ C_{1v} = \exp(-(r - \delta)\tau)K \left[ (N(d_3) - N(d_2)) \frac{\nu}{\omega} - N'(d_2)\sqrt{\bar{v}\tau} \right] \quad (11) \]

where

\[ d_3 = d_2 + \omega \sqrt{\frac{\tau}{\nu}} \]

\( N'(\cdot) \) is the standard normal probability density function. Note that in order to simplify the expression of \( C_{1v} \), the slow moving diffusion part of the return variance has been set to its long term mean, i.e., \( \bar{v} = \frac{\theta}{\kappa} \).
In order to solve Eq (9b), further approximation is needed. The expected change in option price due to the jump of the underlying stock prices is expressed as delta, gamma, vega, and vanna contributions of the jump component to the option price:

\[ E[C(\tau, x + \ln(K) + \ln(1 + J), v) - C(\tau, x + \ln(K), v)] \]

\[ \approx \left( \frac{\partial C_0}{\partial x} \right) \eta + \frac{1}{2} \left( \frac{\partial^2 C_0}{\partial x^2} \right) \eta^2 + \left( \frac{\partial C_0}{\partial \sqrt{v}} \right) \zeta + 2 \left( \frac{\partial^2 C_0}{\partial x \partial \sqrt{v}} \right) \eta \zeta \]

where

\[ \eta = E[\ln(1 + J)] \]

and \( \zeta \) is the return volatility caused by the jump component. If it is assumed that \( J \) follows a log-normal distribution, then \( \eta \) and \( \zeta \) are given in terms jump parameters by (BCC, 1997)

\[ \eta = \ln(1 + \mu_j) - \frac{\sigma_j^2}{2} \]

\[ \zeta = \sqrt{\lambda \left[ \mu_j^2 + \exp(\sigma_j^2) - 1 \right]} \left( 1 + \mu_j \right)^2 \]

With all approximations in place, the Eq (9b) is solved as (useful integral formulas are provided in Appendix)

\[ C_{1J} = \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\tau, \tau', x, x', u, u') f_J(C_0(\tau', x', u')) du' dx' d\tau' \]

\[ C_{1j} = SN(d_1) \frac{1 - \exp(-(r-\delta)\tau)}{r-\delta} \left( \lambda \eta + \frac{\eta^2}{2} - \delta \right) + K \exp(-(r-\delta)\tau) N(d_2) \delta \tau \]

\[ + \frac{\lambda K \exp(-(r-\delta)\tau)}{2 \sqrt{v}} N'(d_2) \left( \eta^2 \sqrt{\tau} - 2d_2 \eta \zeta \tau + \sqrt{\tau} \zeta \sqrt{\tau} \right) \] (12)

Since \( \epsilon \) is an arbitrary constant, the original equation Eq. (4) is recovered from Eq (6) by setting \( \epsilon = 1 \). So does the approximate solution to Eq. (4) from Eq. (7).

In summary, Eqs. (10-12) provide an approximate call option price based on the SVJ model of Eqs. (1) and (2).
II. Brief Summary of Several Commonly Used Option Pricing Models

1. Ad hoc Black-Scholes Model (ad hoc BS)

Dumas, Fleming, and Whaley (1998) presented an ad hoc procedure to smooth Black-Scholes implied volatilities $\sigma_{imp}$ across strike prices $K$ and times to expiration $\tau$:

$$\sigma_{imp} = \max\left\{ 0.01, a_0 + a_1 K + a_2 K^2 + a_3 \tau + a_4 \tau^2 + a_5 K \tau \right\} \quad (13)$$

This simple procedure provides same level of predictive and hedging performance as several other deterministic volatility models, and has been used as a performance baseline to compete with in a few empirical studies.

2. Cumulant Matching Model (CM)

Jarrow and Rudd (1982) pioneered a path-breaking approach to obtain approximate option valuations. Instead of solving the PDE, they employed the generalized Edgeworth series expansion technique to approximate the unknown stock price distribution with simple distributions such as log-normal distribution. Ki et. al. (2005) worked out a closed-form option pricing approximation that expanded Edgeworth series to the fourth moment using a log-normal distribution and satisfied the martingale restriction as the following:

$$C = S N(d_1) - K \exp(-r \tau) N(d_2)$$

$$+ K \exp(-r \tau) \left[ \sigma \sqrt{\tau} E_1 \frac{d N(d_2)}{d d_2} + E_1 \frac{d^2 N(d_2)}{d d_2^2} + E_2 \frac{d^3 N(d_2)}{d d_2^3} \right] \quad (14)$$

where

$$d_1 = \frac{\ln\left( \frac{S}{K} \right) + \left( r + \frac{1}{2} \sigma^2 \right) \tau - \ln\left( 1 + \sigma^2 \tau E_1 \right)}{\sigma \sqrt{\tau}}$$

$$d_2 = d_1 - \sigma \sqrt{\tau}$$

$$E_1 = \frac{1}{6} \xi \sigma \sqrt{\tau} + \frac{1}{24} (\kappa - 3) \sigma^2 \tau$$
\[ E_2 = \frac{1}{24}(\kappa - 3)\sigma \sqrt{\tau} \]

Market data are used to find the implied volatility \( \sigma \), skewness \( \xi \) and kurtosis \( \kappa \) in the above model.

3. Modified Heston Model (Heston)

Heston (1993) model is a widely recognized model. It is parsimonious with five parameters yet it provides much more accurate pricing than the Black-Scholes model. However, the Fourier inversion procedure in Heston's original setting fails to converge to a solution quite often. Kahl and Jäckel (2005) proposed a new calculation scheme to make the Heston model numerically stable for practically any levels of parameters. Their formula is

\[ C = P(\tau) \int_0^1 y(x) \, dx \quad (15) \]

Where \( P(\tau) \) is the discount factor

\[ P(\tau) = \exp(-r\tau) \]

\[ y(x) = \frac{1}{2}(F-K) + \frac{F f_1 \left( -\ln\left( \frac{x}{C_\infty} \right) \right) - K f_2 \left( -\ln\left( \frac{x}{C_\infty} \right) \right)}{x \pi C_\infty} \]

\( F \) is the forward price

\[ F = S \exp(r\tau) \]

\[ f_1(\phi) = \Re \left( \frac{K^{-i\phi} \psi(\phi-i)}{i\phi F} \right) \]

\[ f_2(\phi) = \Re \left( \frac{K^{-i\phi} \psi(\phi)}{i\phi} \right) \]

\[ \psi(\phi) = \exp(E+D\sqrt{v}+i\phi \ln(F)) \]

\[ E = \frac{\kappa \theta}{\sigma_v^2} \left[ (\kappa - \rho \sigma_v \phi + d) \tau - 2 \ln(G) \right] \]

\[ D = \frac{\kappa - \rho \sigma_v \phi + d}{\sigma_v^2} \left( \frac{\exp(d\tau) - 1}{c \exp(d\tau) - 1} \right) \]
\[
\ln(G) = \ln\left(\left|\frac{G_N}{G_D}\right|\right) + i\left[\arg(G_N) - \arg(G_D) + 2\pi(n - m)\right]
\]

\[m = \text{int} \left[\frac{t_c + \pi}{2\pi}\right]\]

\[n = \text{int} \left[\frac{t_c + \Im(d)\tau + \pi}{2\pi}\right]\]

\[G_D = c - 1\]

\[G_N = c \exp(d\tau) - 1\]

\[t_c = \arg(c)\]

\[c = \frac{\kappa - \rho \sigma_v \phi i + d}{\kappa - \rho \sigma_v \phi i - d}\]

\[d = \sqrt{(\rho \sigma_v \phi i - \kappa)^2 + \sigma_v^2(\phi i + \phi_0^2)}\]

\[C_v = \frac{\sqrt{1 - \rho^2}}{\sigma_v}(v + \kappa \theta \tau)\]

In order to numerically carry out the integration in Eq. (15), the limits of \(y(x)\) at the boundaries of the integral are needed:

\[
\lim_{x \to 0} y(x) = \frac{1}{2}(F - K)
\]

\[
\lim_{x \to 1} y(x) = \frac{1}{2}(F - K) + \frac{F \lim_{\phi \to 0} f_1(\phi) - K \lim_{\phi \to 0} f_2(\phi)}{\pi C_v}
\]

\[
\lim_{\phi \to 0} f_1(\phi) = \ln(F/K) + \Im(E'(-i)) + \Im(D'(-i))v
\]

\[
\Im(E'(-i)) = \begin{cases} 
0 \kappa \exp(- (\kappa - \rho \sigma_v)\tau) + \theta \kappa ((\kappa - \rho \sigma_v)\tau - 1) & , \kappa \neq \rho \sigma_v \\
\frac{2(\kappa - \rho \sigma_v)^2}{\kappa \theta \tau^2} & , \kappa = \rho \sigma_v
\end{cases}
\]
\[ \mathcal{Z}(D'(-i)) = \begin{cases} \frac{1 - \exp(-\kappa - \rho \sigma_v \tau)}{2(\kappa - \rho \sigma_v)}, & \kappa \neq \rho \sigma_v \\ \frac{\tau}{2}, & \kappa = \rho \sigma_v \end{cases} \]

\[ \lim_{\phi \to 0} f_{2}(\phi) = \ln(F/K) + \mathcal{Z}(E'(0)) + \mathcal{Z}(D'(0))v \]

\[ \mathcal{Z}(E'(0)) = -\frac{\theta \kappa \exp(-\kappa \tau) + \theta \kappa (\kappa \tau - 1)}{2\kappa^2} \]

\[ \mathcal{Z}(D'(0)) = -\frac{1 - \exp(-\kappa \tau/2)}{2\kappa} \]

Furthermore, the term \( \exp(d \tau) \) tends to explode for certain sets of parameter values. In these cases an arbitrarily large number can be used to replace it.

4. BCC Model (BCC SVJ)

Bakshi, Cao, and Chen (1997) expanded Heston's approach to include jump and stochastic interest rate in the model. To simplify the presentation, only the result of the SVJ model is presented here:

\[ C = S \Pi_1 - K \exp(-r \tau) \Pi_2 \quad (16) \]

where

\[ \Pi_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{K^{-i \phi} f_j}{i \phi} \right] d\phi \]

\[ f_1 = \exp \{ i \phi r \tau - \frac{\theta}{\sigma_v^2} 2 \ln \left( 1 - \left[ \frac{\xi - \kappa + (1 + i \phi) \rho \sigma_v}{2 \xi} \right] \frac{1 - \exp(-\xi \tau)}{2 \xi} \right) \]

\[ -\frac{\theta}{\sigma_v^2} [\xi - \kappa + (1 + i \phi) \rho \sigma_v] \tau + i \phi \ln(S) \]

\[ +\lambda (1 + \mu_j) \tau \left[ (1 + \mu_j)^v \exp((i \phi/2)(1 + i \phi) \sigma_j^2) - 1 \right] - \lambda i \phi \mu_j \tau \]

\[ + \frac{i \phi(i \phi + 1)(1 - \exp(-\xi \tau))}{2 \xi - [\xi - \kappa + (1 + i \phi) \rho \sigma_v](1 - \exp(-\xi \tau))} \]

\[ f_2 = \exp \{ i \phi r \tau - \frac{\theta}{\sigma_v^2} 2 \ln \left( 1 - \left[ \frac{\xi^* - \kappa + i \phi \rho \sigma_v}{2 \xi^*} \right] \frac{1 - \exp(-\xi^* \tau)}{2 \xi^*} \right) \]

\[ Eola Investments LLC All Rights Reserved \]
\[-\frac{\theta}{\sigma_v^2} \left[ \xi^* - \kappa + i \phi \rho \sigma_v \right] \tau + i \phi \ln(S) + \lambda \tau \left[ (1 + \mu_j)^i \exp((i \phi/2)(i \phi - 1) \sigma_j^2) - 1 \right] - \lambda i \phi \mu_j \tau \]
\[+ \frac{i \phi(i \phi - 1)(1 - \exp(-\xi^* \tau))}{2 \xi^* - [\xi^* - \kappa + i \phi \rho \sigma_v](1 - \exp(-\xi^* \tau))} v \}

\[\xi^* = \sqrt{[\kappa - (1 + i \phi) \rho \sigma_v]^2 - i \phi(i \phi + 1) \sigma_v^2} \]
\[\xi = \sqrt{[\kappa - (1 + i \phi) \rho \sigma_v]^2 - i \phi(i \phi + 1) \sigma_v^2} \]

Interestingly, adding the jump terms in the characteristic function makes the inverse Fourier transformation much more stable compared to the Heston model.

5. Heston-Nandi GARCH Option Model (HN)

Heston and Nandi (2000) presented a (semi) closed-form solution to a GARCH potion model with Gaussian innovations. The one-period lag GARCH process they considered in a risk-neutral measure is the following

\[\ln\left( \frac{S_t}{S_{t-1}} \right) = r - \frac{v_t}{2} + \sqrt{v_t} z_t; \quad z_t \sim N(0,1) \]
\[v_t = \omega^* + \beta^* v_{t-1} + \alpha^* (z_{t-1} - \gamma^* \sqrt{v_{t-1}})^2 \quad (17)\]

For the above GARCH process to be meaningful, there are two constrains to the model parameters:

\[\omega^* \geq 0 \quad \text{and the persistency} \quad \beta^* + \alpha^* \gamma^2 < 1 .\]

The European call option price is given by

\[C = \exp(-r \tau) f(1) \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{K^{-i \phi} f(i \phi + 1)}{i \phi f(1)} \right] d \phi \right) \]
\[- \exp(-r \tau) K \left( \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{K^{-i \phi} f(i \phi)}{i \phi} \right] d \phi \right) \quad (18)\]

where

\[f(\phi) = S_t^{\phi} \exp(A_t + B_t v_{t+1})\]
The coefficients $A$ and $B$ are not dependent upon the spot price and strike prices. They only depend on time to expiration. Their values are calculated recursively backwards starting from the time at expiration, $T$:

$$A_T = B_T = 0$$

$$A_t = A_{t+1} + \phi r + B_{t+1} \omega^* - \ln \left(1 - 2 \alpha^* \right)/2$$

$$B_t = \phi \left( y^* - \frac{1}{2} \right) - \frac{1}{2} y^{*2} + \beta^* B_{t+1} + \frac{(\phi - y^*)^2}{2 - 4 \alpha^* B_{t+1}}$$

In theory the spot return variance $v_{t+1}$ is predicted by the GARCH process. In practice, however, it is commonly replaced by its expected value

$$E(v_{t+1}) = \omega^* + \beta^* E(v_t) + \alpha^* E \left( (z_t - y^* \sqrt{v_t})^2 \right)$$

$$E(v_{t+1}) = \frac{\omega^* + \alpha^*}{1 - \beta^* - \alpha^* y^{*2}}$$

III. Empirical test

An empirical test was conducted with the market prices of S&P 500 European options (SPX) obtained from CBOE website each day after market close. Three days in early 2010 were selected to test the in-sample fitting, namely January 11th, 21st, and February 3rd. Their three consecutive days (January 12th, 22nd and February 4th) were selected to test the out-of-sample forecasting using previous day's parameters. These days were selected based on the S&P 500 index daily returns which were -1%, -2%, and -3%, respectively.

The middle point of the bid-ask spreads were taken as the market option prices. Deep out of money options with 0 bid price were excluded. Since the minimum tick is 0.05 for options trading below 3.00 and 0.10 for all other series, there were options having same prices but with different strikes. In these cases, only the calls with the lowest strike prices and the puts with the highest strike
prices were retained. The spot index values were corrected by subtracting the present values of the expected future dividends within the contractual periods, which were obtained from the put and call prices and the interest rates with matching time-to-expiration through put-call parity relation. The risk-free interest rates were obtained from the website of Federal Reserve Bank of St. Louis. Option prices that appeared to violate the concave curvature requirement due to different trading times were also omitted in the study.

Only the market call prices were used to fit the option models, since the market put prices had been used to obtain the dividend correction already. In other words, put-call parity ensures that the results and conclusions obtained from tests on call options will apply to put options.

The results of the empirical test are presented in Table 1. The accuracy of in-sample fit and out of sample forecast is given by the Root Mean Square Error (RMSE) values, which are calculated by

\[ RMSE = \sqrt{\frac{\sum (model \ price - market \ price)^2}{number \ of \ options}}. \]

The mean of the bid-ask spread (MBAS) is also listed in the table to estimate the uncertainty about market prices. The calculation time for individual options is not as important as the time required for model calibration, which is typically the most time consuming step in practical applications. However, the time for model calibration is affected by what initial values to choose. In the empirical study, the initial values for in-sample fit were chosen by trial and error within a reasonable range. The values of the fitting parameters were those that gave the smallest RMSE, and were used to obtain the out-of-sample forecasting for the consecutive day. After they being used to obtain out-of-sample calculation, the previous day's parameters were also used as the initial values to calibrate the models and the CPU times were recorded. Presented in Table 1 were the CPU times normalized by the time for calibrating the Heston model using the modified procedure.

The test results show that the first order approximation of SV model (Eqs. 10 and 11) is not quite accurate; it only out-performs the ad hoc BS model. However, adding the first order approximation of jump process (Eq. 12) improves accuracy significantly. The approximate SVJ model
clearly out-performs the Cumulant Matching model, and only slightly under-performs Heston-Nandi GARCH model. But the approximate SVJ model is three orders of magnitude faster than the Heston-Nandi GARCH model.

The approximate formula can be further improved by adding interest rate term structures. BCC (1997) implied that as long as the correlation between the interest rate process and stock price process or the volatility process can be neglected, adding the interest rate stochastic process only modifies the discount factor in the option model. One could choose whatever bond pricing model to improve the approximate SVJ option model. Here I used a simple descriptive 3-factor model to represent the interest rate term structure,

\[ r = a_0 + a_1 \tau + a_2 \tau^2 \]  (19)

where \( a_0, a_1, \) and \( a_2 \) are level, slope, and curvature factors, respectively. Adding the three fitting parameters in the approximate SVJ model further improves its accuracy without adding more computation burden thanks to the simple closed-form formula. The empirical test results of the approximate SVJ model with interest rate term structure in Table 1 (Approx. SVJ + IRTS) show that its accuracy now is the same as or even slightly better than those of the Heston model. And it is one to two orders of magnitude faster than the Heston model.

Success in fitting the market data does not mean the function is really a good approximation to Eq. 3, without any discussion about the error of approximation. It could be a serendipitous fitting function that has nothing to do with the solution to Eq. 3. I show below using one numerical example the accuracy of the approximation as compared to the Black-Scholes equation. The exact solution of BCC (1997) to Eq. 3 enables the calculation of the sum of squared errors for both SVJ approximation (Eqs. 10, 11, 12 and 19) and Black-Scholes equation. The average values of the BCC SVJ parameters in the 6 days discussed before are \( \nu = 0.0417, \ \theta = 0.0597, \ \kappa = 1.2870, \ \sigma_v = 0.4441, \ \rho = -0.6906, \ \lambda = 0.0661, \ \mu_J = -0.4028, \ \sigma_J = 0.2372. \) These values were used to calculate call option
values with different time-to-expiration and moneyness. The call option values normalized by strike prices as a function of time-to-expiration and moneyness are tabulated in Table 2, which show that the approximation is good for in-the-money options and long dated options. As such, in the whole range of moneyness from 0.8 to 1.2 and time-to-expiration from 5 days to 3 years the sum of squared error of the approximated solution is only 14% of that of the Black-Scholes equation.

The values of the parameters fitted from the approximate SV solution (Eqs. 10 and 11) and the approximate SVJ solution (Eqs. 10, 11, 12 and 19) using the above discussed six days' market data are also very reasonable. Unconstrained minimization of RMSE provides negative correlation and mean jump size values and positive variance and jump frequency values everyday. The average values of the parameters in the six days are \( \bar{\nu} = 0.0355, \nu = 0.0536, \) and \( \rho \sigma_v = -0.1154 \) for the approximated SV solution and \( \bar{\nu} = 0.0457, \nu = 0.0477, \rho \sigma_v = -0.2721, \lambda = 0.9522, \mu_J = -0.0263, \) and \( \sigma_J = 0.0436 \) for the approximated SVJ solution.

IV. Conclusions

In order to obtain a fast converging closed-form option pricing formula, a rather coarse first order regular perturbation approximation was employed to solve the pricing PDE. The combination of coarse approximation and more realistic SVJ model with interest rate term structures results in a option pricing formula for European options that provides the same accuracy as, but one to two orders of magnitude faster than, the Heston model. A natural extension in future work is to use the same first order regular perturbation and Green's function approach to solve the early excise premium for American put options.
Appendix: Several Integral Formulas for Deriving the Approximate Solution

\[
\int_0^\infty \int_{-\infty}^\infty G(\tau, \tau', x, x', u, u') \exp(x') N(d_1(\tau', x')) du' dx' d\tau' = \exp(x) N(d_1) \frac{1 - \exp(-(r-\delta)\tau)}{r-\delta} \tag{A1}
\]

\[
\int_0^\infty \int_{-\infty}^\infty G(\tau, \tau', x, x', u, u') \exp(-(r-\delta)\tau') N(d_2(\tau', x')) du' dx' d\tau' = \tau N(d_2) \tag{A2}
\]

\[
\int_0^\infty \int_{-\infty}^\infty G(\tau, \tau', x, x', u, u') \exp(-(r-\delta)\tau') N'(d_2(\tau', x')) \sqrt{\tau'} du' dx' d\tau' = \frac{1}{2} N'(d_2) \sqrt{\tau} \tag{A3}
\]

\[
\int_0^\infty \int_{-\infty}^\infty G(\tau, \tau', x, x', u, u') \frac{\exp(-(r-\delta)\tau') N'(d_2(\tau', x'))}{\sqrt{\tau'}} du' dx' d\tau' = N'(d_2) \sqrt{\tau} \tag{A4}
\]

\[
\int_0^\infty \int_{-\infty}^\infty G(\tau, \tau', x, x', u, u') \exp(-(r-\delta)\tau') d_2(\tau', x') N'(d_2(\tau', x')) \tau' du' dx' d\tau' = \frac{1}{3} d_2 N'(d_2) \tau^2 \tag{A5}
\]

\[
\int_0^\infty \int_{-\infty}^\infty G(\tau, \tau', x, x', u, u') \exp(-(r-\delta)\tau') d_2^2(\tau', x') N'(d_2(\tau', x')) \sqrt{\tau'} du' dx' d\tau' = \frac{1}{3} N'(d_2) \sqrt{\tau^3} \left(\frac{1}{2} + d_2^2\right) \tag{A6}
\]
\[ \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty G(\tau, \tau', x, x', u, u') \exp(-(r-\delta)\tau') \, d_2(\tau', x') \, N'(d_2(\tau', x')) \, du \, dx \, d\tau' \]

\[ = \frac{1}{2} d_2 N'(d_2) \tau \quad (A7) \]
Table 1. Empirical results of in-sample fitting and out-of-Sample forecasting of S&P 500 European (SPX) call options. MBAE stands for mean bid-ask spread; RMSE root mean square error, CM cumulant matching model, BCC SVJ Bakshi, Cao, and Chen (1997) stochastic-volatility with jump model; HN Heston and Nandi (2000) GARCH model; Approx. SV Eqs.10 and 11; Approx. SVJ Eqs. 10, 11, and 12; Approx. SVJ+IRTS Eqs. 10, 11, 12 and 19. The CPU time is normalized by the time spent on calibrating the Heston model.

<table>
<thead>
<tr>
<th>Date</th>
<th>01/11/10</th>
<th>01/12/10</th>
<th>01/21/10</th>
<th>01/22/10</th>
<th>02/03/10</th>
<th>02/04/10</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>1146.98</td>
<td>1136.22</td>
<td>1116.48</td>
<td>1091.76</td>
<td>1097.28</td>
<td>1063.11</td>
</tr>
<tr>
<td>Daily Return</td>
<td>–</td>
<td>-0.94%</td>
<td>–</td>
<td>-2.21%</td>
<td>–</td>
<td>-3.11%</td>
</tr>
<tr>
<td>Number of Options</td>
<td>577</td>
<td>608</td>
<td>649</td>
<td>652</td>
<td>577</td>
<td>643</td>
</tr>
<tr>
<td>MBAS</td>
<td>4.12</td>
<td>3.20</td>
<td>3.58</td>
<td>4.16</td>
<td>3.65</td>
<td>3.58</td>
</tr>
<tr>
<td>Test Type</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ad hoc BS</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>5.61</td>
<td>5.60</td>
<td>7.35</td>
<td>6.73</td>
<td>5.58</td>
<td>4.55</td>
</tr>
<tr>
<td>CPU Time</td>
<td>–</td>
<td>2.7E-4</td>
<td>–</td>
<td>2.6E-4</td>
<td>–</td>
<td>2.8E-4</td>
</tr>
<tr>
<td>CM</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>4.42</td>
<td>4.03</td>
<td>3.27</td>
<td>4.20</td>
<td>3.29</td>
<td>4.20</td>
</tr>
<tr>
<td>CPU Time</td>
<td>–</td>
<td>5.2E-3</td>
<td>–</td>
<td>6.1E-3</td>
<td>–</td>
<td>5.3E-3</td>
</tr>
<tr>
<td>Heston</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>1.53</td>
<td>1.66</td>
<td>1.43</td>
<td>3.64</td>
<td>1.45</td>
<td>3.95</td>
</tr>
<tr>
<td>CPU Time</td>
<td>–</td>
<td>1.0E0</td>
<td>–</td>
<td>1.0E0</td>
<td>–</td>
<td>1.0E0</td>
</tr>
<tr>
<td>BCC SVJ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>1.14</td>
<td>1.25</td>
<td>1.07</td>
<td>3.51</td>
<td>1.03</td>
<td>3.85</td>
</tr>
<tr>
<td>CPU Time</td>
<td>–</td>
<td>2.7E0</td>
<td>–</td>
<td>5.8E0</td>
<td>–</td>
<td>5.5E0</td>
</tr>
<tr>
<td>HN</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>3.40</td>
<td>3.17</td>
<td>2.05</td>
<td>3.70</td>
<td>2.21</td>
<td>4.02</td>
</tr>
<tr>
<td>CPU Time</td>
<td>–</td>
<td>2.1E1</td>
<td>–</td>
<td>1.6E1</td>
<td>–</td>
<td>5.4E1</td>
</tr>
<tr>
<td>Approx. SV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>4.98</td>
<td>4.67</td>
<td>4.24</td>
<td>5.36</td>
<td>4.17</td>
<td>5.05</td>
</tr>
<tr>
<td>CPU Time</td>
<td>–</td>
<td>4.2E-3</td>
<td>–</td>
<td>3.7E-3</td>
<td>–</td>
<td>3.8E-3</td>
</tr>
<tr>
<td>Approx. SVJ</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>2.29</td>
<td>2.36</td>
<td>2.26</td>
<td>4.23</td>
<td>2.17</td>
<td>4.30</td>
</tr>
<tr>
<td>CPU Time</td>
<td>–</td>
<td>2.6E-2</td>
<td>–</td>
<td>2.2E-2</td>
<td>–</td>
<td>2.1E-2</td>
</tr>
<tr>
<td>Approx. SVJ+IRTS</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RMSE</td>
<td>1.39</td>
<td>1.51</td>
<td>1.37</td>
<td>3.65</td>
<td>1.18</td>
<td>3.94</td>
</tr>
<tr>
<td>CPU Time</td>
<td>–</td>
<td>6.5E-2</td>
<td>–</td>
<td>3.4E-2</td>
<td>–</td>
<td>5.2E-2</td>
</tr>
</tbody>
</table>
Table 2. An numerical example for the accuracy of the SVJ approximation as compared to the exact solution and the Black-Scholes call option value. The values of the call options are normalized by the strike price. **BCC SVJ** stands for the exact solution of the Bakshi, Cao, and Chen (1997) stochastic-volatility with jump model; **Approx. SVJ** Eqs. 10, 11, 12 and 19. **BS** Black-Scholes call price. \( \nu = 0.0417, \quad \theta = 0.0597, \quad \kappa = 1.2870, \quad \sigma_s = 0.4441, \quad \rho = -0.6906, \quad \lambda = 0.0661, \quad \mu_J = -0.4028, \quad \sigma_J = 0.2372, \quad \tau = 0.0300 \)

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\text{Moneyness} & \text{Approx. SVJ} & \text{BS} & \text{Moneyness} & \text{Approx. SVJ} & \text{BS} \\
\text{(Day)} & S/K & \text{BCC SVJ} & \text{BCC SVJ} & \text{BS} & \text{BCC SVJ} & \text{BS} \\
\hline
5 & 0.80 & 0.000000 & 0.000000 & 0.000000 & 126 & 0.80 & 0.000310 & 0.000382 & 0.001626 \\
5 & 0.85 & 0.000000 & 0.000000 & 0.000000 & 126 & 0.85 & 0.002063 & 0.004383 & 0.005351 \\
5 & 0.90 & 0.000000 & 0.000000 & 0.000000 & 126 & 0.90 & 0.009412 & 0.014977 & 0.013794 \\
5 & 0.95 & 0.000070 & 0.000292 & 0.000139 & 126 & 0.95 & 0.027446 & 0.034251 & 0.029216 \\
5 & 1.00 & 0.009909 & 0.011861 & 0.009740 & 126 & 1.00 & 0.056056 & 0.062291 & 0.052918 \\
5 & 1.05 & 0.051035 & 0.051052 & 0.050588 & 126 & 1.05 & 0.092158 & 0.097249 & 0.084739 \\
5 & 1.10 & 0.100726 & 0.100421 & 0.100414 & 126 & 1.10 & 0.133113 & 0.136764 & 0.123312 \\
5 & 1.15 & 0.150698 & 0.150418 & 0.150411 & 126 & 1.15 & 0.177151 & 0.179054 & 0.166754 \\
5 & 1.20 & 0.200674 & 0.200418 & 0.200411 & 126 & 1.20 & 0.223143 & 0.223186 & 0.213295 \\
21 & 0.80 & 0.000000 & 0.000000 & 0.000000 & 252 & 0.80 & 0.003219 & 0.004866 & 0.008527 \\
21 & 0.85 & 0.000001 & -0.000001 & 0.000006 & 252 & 0.85 & 0.010530 & 0.015441 & 0.017429 \\
21 & 0.90 & 0.000068 & 0.000383 & 0.000287 & 252 & 0.90 & 0.026410 & 0.033079 & 0.031508 \\
21 & 0.95 & 0.003013 & 0.005418 & 0.005884 & 252 & 0.95 & 0.051424 & 0.058014 & 0.051530 \\
21 & 1.00 & 0.021047 & 0.024671 & 0.020396 & 252 & 1.00 & 0.083506 & 0.089419 & 0.077663 \\
21 & 1.05 & 0.057793 & 0.059568 & 0.055658 & 252 & 1.05 & 0.120488 & 0.125864 & 0.109517 \\
21 & 1.10 & 0.104012 & 0.103666 & 0.102183 & 252 & 1.10 & 0.160832 & 0.165818 & 0.146310 \\
21 & 1.15 & 0.153106 & 0.152044 & 0.151754 & 252 & 1.15 & 0.203516 & 0.208017 & 0.187081 \\
21 & 1.20 & 0.202853 & 0.201783 & 0.201726 & 252 & 1.20 & 0.247863 & 0.251601 & 0.230867 \\
63 & 0.80 & 0.000009 & -0.000071 & 0.000122 & 756 & 0.80 & 0.039947 & 0.038174 & 0.046479 \\
63 & 0.85 & 0.000197 & 0.000683 & 0.000969 & 756 & 0.85 & 0.063945 & 0.062124 & 0.065656 \\
63 & 0.90 & 0.002446 & 0.005343 & 0.004681 & 756 & 0.90 & 0.092883 & 0.090975 & 0.088738 \\
63 & 0.95 & 0.013476 & 0.018754 & 0.015225 & 756 & 0.95 & 0.125683 & 0.124095 & 0.115582 \\
63 & 1.00 & 0.038121 & 0.043470 & 0.036393 & 756 & 1.00 & 0.161461 & 0.160772 & 0.145932 \\
63 & 1.05 & 0.073778 & 0.077686 & 0.068943 & 756 & 1.05 & 0.199553 & 0.200303 & 0.179461 \\
63 & 1.10 & 0.116169 & 0.117926 & 0.110301 & 756 & 1.10 & 0.239467 & 0.242044 & 0.215807 \\
63 & 1.15 & 0.162212 & 0.161840 & 0.156789 & 756 & 1.15 & 0.280838 & 0.285445 & 0.254598 \\
63 & 1.20 & 0.210120 & 0.208290 & 0.205605 & 756 & 1.20 & 0.323395 & 0.330057 & 0.295476 \\
\hline
\end{array}
\]

Sum of Squared Error of Approx. SVJ = 0.000604

Sum of Squared Error of BS = 0.004396
REFERENCES


